Engineering Notes

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Some Remarks on the Solution of the Lifting Line Equation

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Introduction

SO much has been said about Prandtl's lifting line equation that it seems futile to elaborate on it further. However, the author feels that a few salient features of this equation have not been remarked upon earlier. The equation is[1]

$$\Gamma(y) = \pi U C(y)[\alpha(y) - \epsilon(y)] \tag{1}$$

where

$$\epsilon(y) = \frac{1}{4\pi U}\int_{-s}^{s}\frac{d\Gamma}{d\eta}\times\frac{d\eta}{(y-\eta)} \tag{2}$$

and

$$\Gamma(s) = \Gamma(-s) = 0 \tag{3}$$

Here Γ is the local circulation, α the local geometric angle of attack measured from the zero lift line, C the local chord, ϵ the local downwash induced by the trailing vortices, U the freestream velocity, s the wing semispan, and y the spanwise coordinate.

Equation (1) is usually solved using a sine series for Γ in the form

$$\Gamma(\theta) = \sum_{n=1}^{\infty} A_n\Gamma_n(\theta) = 2sU\sum_{n=1}^{\infty} A_n \sin n\theta \tag{4}$$

in a collocation method, and $\theta = \cos^{-1}(y/s)$.

Since Γ and α are related by a linear operator, it may be shown that

$$\alpha(\theta) = \sum_{n=1}^{\infty} A_n\alpha_n(\theta) = \frac{2s}{\pi C(y)}\sum_{n=1}^{\infty} A_n \sin n\theta + \epsilon(\theta) \tag{5}$$

and

$$\epsilon(\theta) = \sum_{n=1}^{\infty} A_n\epsilon_n(\theta) = \frac{1}{2}\sum_{n=1}^{\infty} A_n n \sin n\theta/\sin\theta \tag{6}$$

The lift and drag coefficients are, respectively,

$$C_L = \pi AR A_1/2 \tag{7}$$

$$C_D = \pi AR\sum_{n=1}^{\infty} nA_n^2/4 \tag{8}$$

where the aspect ratio $AR = 4s^2/S$, and S is the wing area.

We note that C_D contains only the square of the unknown constants A_n. In other words, the loadings (Γ_n, α_n) are orthogonal in the sense used by Graham.[3] Orthogonality is usually associated with simplicity, and this is reflected in Eqs. (7) and (8).

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Let us assume that a loading (Γ, α) is expressible by a finite number of terms m of Eqs. (4) and (5) to the desired accuracy. Then from the orthogonality of the loadings it may be easily shown that

$$\begin{aligned}\int_{-s}^{s}\Gamma(y)\epsilon_n(y)dy &= \int_{-s}^{s}\epsilon(y)\Gamma_n(y)dy \\ &= A_n\int_{-s}^{s}\Gamma_n(y)\epsilon_n(y)dy \\ &= A_n\frac{\pi}{2}s^2Un\end{aligned} \tag{9}$$

for $n = 1, \ldots, m$.

Hence if the circulation distribution is given, the unknown constants may be obtained from

$$A_n = 2\int_{-s}^{s}\Gamma(y)\epsilon_n(y)dy/(\pi s^2Un) \tag{10}$$

to yield $\alpha(y)$ from Eqs. (5) and (6). If the downwash distribution $\epsilon(y)$ is given, the constants may be obtained from

$$A_n = 2\int_{-s}^{s}\epsilon(y)\Gamma_n(y)dy/(\pi s^2Un) \tag{11}$$

to yield $\Gamma(y)$ from Eq. (4). However, more frequently the wing geometry is given in which case $\alpha(y)$ is specified. Using Eq. (5) to substitute for $\epsilon(y)$ in Eq. (11) results in m linear simultaneous equations

$$\begin{aligned}A_n\frac{\pi}{2}s^2Un + (1/\pi U)\int_{-s}^{s}\Big[\Gamma_n(y)\sum_{r=1}^{m}A_r\Gamma_r(y)/C(y)\Big]dy \\ = \int_{-s}^{s}\alpha(y)\Gamma_n(y)dy\end{aligned} \tag{12}$$

which may be solved for the A_n. Results of Eqs. (10) and (11) are important since they allow the A_n to be evaluated explicitly. Equation (12) clearly shows that for elliptic planforms, i.e. $C(\theta) \sim \sin\theta$, the A_n can be obtained explicitly for any angle of attack distribution. In this form it is a generalization of Filotas' results.[2] It is further interesting to note that Eq. (12) is the same as would be obtained by an application of the Galerkin method[4] to Eq. (1). The present derivation shows that the Galerkin approach is a very natural one and is the reason for its success in Ref. 4.

The advantage of using Eqs. (10–12) is that the solution does not depend only on the characteristics of the wing at a small number of isolated points as in a collocation procedure, but gives an approximate solution along the entire span. Hence discontinuities, flap deflections, etc., may be accounted for.

An Example—Rectangular Wing

Application of Eq. (12) to a wing of constant chord C, and constant incidence α, gives

$$\frac{\pi}{2}nA_n - \frac{16s}{\pi C}\sum_{r=1}^{m}{}' \frac{nrA_r}{(r^2-n^2)^2 - 2(r^2+n^2)+1} \begin{cases} = \pi\alpha\,; & n = 1 \\ = 0\,; & n \neq 1 \end{cases} \tag{13}$$

for $n = 1, \ldots, m$;

where Σ' denotes summation over only those terms for which $(n + r)$ is even. A one term approximation for Γ yields

$$A_1 = \pi\alpha/(\pi/2 + 16s/3\pi C) \tag{14}$$

A two term approximation shows

$$\begin{aligned} A_1 &= \pi\alpha\Big/\left[\left(\pi/2 + \frac{16s}{3\pi C}\right) - \left(\frac{16s}{15\pi C}\right)^2\Big/\left(\frac{\pi}{2} - \frac{144s}{35\pi C}\right)\right] \\ A_2 &= 0 \\ A_3 &= \frac{16s\alpha}{15C}\Big/\left[\left(\frac{\pi}{2} + \frac{16s}{3\pi C}\right)\left(\frac{\pi}{2} - \frac{144s}{35\pi C}\right) - \left(\frac{16s}{15\pi C}\right)^2\right] \end{aligned} \tag{15}$$

The computation effort required in solving Eq. (13) is less compared to a collocation method where trigonometric functions must be evaluated.

Conclusions

The method outlined in this note is valid for any set of loadings (Γ_n, α_n) which are orthogonal in Graham's sense. It may be used for non-orthogonal loadings if they are first converted to an orthogonal set as suggested by Graham.[3]

References

[1]Glauert, H., "The Elements of Aerofoil and Airscrew Theory," Cambridge University Press, New York, 1926.

[2]Filotas, L. T., "Solution of the Lifting Line Equation for Twisted Elliptic Wings," *Journal of Aircraft,* Vol. 8, No. 10, Oct. 1971, pp. 835–836.

[3]Graham, E. W., "A Drag Reduction Method for Wings of Fixed Planform," *Journal of the Aeronautical Sciences,* Vol. 19, No. 12, 1952, pp. 823–825.

[4]Anderson, R. C. and Millsaps, K., "Application of the Galerkin Method to the Prandtl Lifting Line Equation," *Journal of Aircraft,* Vol. 1, No. 3, May–June 1964, pp. 126–128.

Generalization of Dunkerley's Formula for Finding the Lowest Critical Velocity of Rotating Shafts

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THE well-known Dunkerley's method is widely used for finding an approximate value of the first critical velocity of forward precession of rotating shafts.[1-3] It is based on the mathematical property of algebraic equations such that if

$$\alpha_0\lambda^n + \alpha_1\lambda^{n-1} + \ldots + \alpha_{n-1}\lambda + \alpha_n = 0 \tag{1}$$

is an algebraic equation, then

$$\sum_{i=1}^{n}\lambda_i = -\frac{\alpha_1}{\alpha_0} \tag{2}$$

In the case of critical velocities of a rotating shaft with n masses, when the damping effects and the weight of the shaft are neglected, the characteristic Eq. (1) is given by

$$\begin{vmatrix} a_{11}m_1 - \lambda & a_{12}m_2 & \cdots & a_{1n}m_n \\ a_{21}m_1 & a_{22}m_2 - \lambda & \cdots & a_{2n}m_n \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}m_1 & a_{n2}m_2 & \cdots & a_{nn}m_n - \lambda \end{vmatrix} = 0 \tag{3}$$

where m_i are the concentrated masses, a_{ik} the flexibility factors and the critical velocities ω_i are given by $\omega = (1/\lambda)^{1/2}$. Equation (2) now is written

$$\sum_{i=1}^{n}\frac{1}{\omega_i^2} = \sum_{i=1}^{n}\lambda_i = -\frac{\alpha_1}{\alpha_0} = \sum_{i=1}^{n}a_{ii}m_i = \sum_{i=1}^{n}\frac{1}{\bar{\omega}_i^2}, \tag{4}$$

where $\bar{\omega}_i^2$ is the square of the critical velocity of the shaft with the mass m_i alone.

Because $\omega_1 \ll \omega_2 \ll \cdots \omega_n$, it's possible to write[1,2]

$$\frac{1}{\omega_1^2} \cong \sum_{i=1}^{n}\frac{1}{\omega_i^2} = \sum_{i=1}^{n}\frac{1}{\bar{\omega}_i^2} \tag{5}$$

This is Dunkerley's formula in the usual form. It gives a very good approximation of the first critical velocity when the lateral inertia of the masses is not important.[2,3]

When the lateral inertia of m of the masses is considerable, the characteristic Eq. (1) is given by

$$\begin{vmatrix} a_{11}m_1 - \lambda & a_{12}m_2 & \cdots & a_{1n}m_n & -b_{11}D_1 & -b_{12}D_2 & \cdots & -b_{1m}D_m \\ a_{21}m_1 & a_{22}m_2 - \lambda & \cdots & a_{2n}m_n & -b_{21}D_1 & -b_{22}D_2 & \cdots & -b_{2m}D_m \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1}m_1 & a_{n2}m_2 & \cdots & (a_{nn}m_n - \lambda) & -b_{n1}D_1 & -b_{n2}D_2 & \cdots & -b_{nm}D_m \\ a_{11}'m_1 & a_{12}'m_2 & \cdots & a_{1n}'m_n & (-b_{11}'D_1 - \lambda) & -b_{12}'D_2 & \cdots & -b_{1m}'D_m \\ a_{21}'m_1 & a_{22}'m_2 & \cdots & a_{2n}'m_n & -b_{21}'D_1 & (-b_{22}'D_2 - \lambda) & \cdots & -b_{2m}'D_m \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1}'m_1 & a_{m2}'m_2 & \cdots & a_{mn}'m_n & -b_{m1}'D_1 & -b_{m2}'D_2 & \cdots & -b_{mm}'D_m - \lambda \end{vmatrix} = 0 \tag{6}$$

where a_{ik}, b_{ik}, a'_{ik}, b'_{ik} are flexibility factors and D_k are the differences between the moment of inertia of each mass with respect to the axis tangent to the elastic line and the moment of inertia with respect to the axis perpendicular to the elastic line.[2] If all the D_k are positive (that is, if there are m discs and $n - m$ concentrated masses), Eq. (6) has n real positive roots and m real negative roots.[2,5]

In this case Eq. (2) is written, taking into account the lateral inertia

$$\sum_{i=1}^{n+m}\frac{1}{\omega_i^2} = \sum_{i=1}^{n}a_{ii}m_i + \sum_{i=1}^{m}(-b_{ii}'D_i) = \sum_{i=1}^{n+m}\frac{1}{\bar{\omega}_i^2} \tag{7}$$

where $\bar{\omega}_i^2$ is now the square of the critical velocity of the shaft with the mass m_i alone (positive) or with the lateral inertia D_i alone (negative).

If $|\omega_1^2| \ll |\omega_2^2|$ it's possible to write

$$\frac{1}{\omega_1^2} \cong \sum_{i=1}^{n+m}\frac{1}{\omega_i^2} = \sum_{i=1}^{n+m}\frac{1}{\bar{\omega}_i^2} \tag{8}$$

This formula gives a very good approximation of the first critical velocity of a rotating shaft except when $|\omega_1^2| \simeq |\omega_2^2|$, that is, when the first real and the first imaginary critical speeds are similar in modulus, a situation which also causes the classical iterative methods to fail.[4]

Example

As an example, a real shaft with two supports was analyzed. For the computation, the real shaft was subdivided into ten sections, and for two of them (representing compressor and turbine), the lateral inertia was taken into account. The supports were at the left end of the shaft and between the 6th and 7th section.

The compressor was substituted by the 3rd section and the turbine by the 10th section. The span between the two supports was 24.5 cm and the overspan between the sec-

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